



ELSEVIER

Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

Linear Algebra and its Applications 392 (2004) 45–59

LINEAR ALGEBRA
AND ITS
APPLICATIONSwww.elsevier.com/locate/laa

A vector-bundle version of a theorem of V. Doležal

Kevin A. Grasse

Department of Mathematics, University of Oklahoma, Norman, OK 73019, USA

Received 12 September 2002; accepted 26 May 2004

Submitted by P. Van Dooren

Abstract

A well known theorem of Doležal states that given a square matrix $A(t)$ that is a C^k function of the scalar parameter $t \in \mathbb{R}$ and whose rank is constant as the parameter varies, one can span the range and kernel of $A(t)$ by linearly independent vectors that are also C^k functions of t . Contemporaneously with (and independently of) the development of Doležal's theorem, Sibuya, individually and in joint work with Hsieh, proved related results on parametric matrix decompositions for parameters in \mathbb{R}^n or \mathbb{C}^n , from which Doležal's theorem can be easily extended to matrices depending on parameters in a rectangular subset of a finite-dimensional Euclidean space. In this paper we explore further generalizations of Doležal's theorem in which the parameter space \mathbb{R} is replaced by a topological space when $k = 0$ or a differentiable manifold when $k \geq 1$. We show that for a broad class of parameter spaces one can associate two vector bundles to a constant-rank, C^k parameterized matrix function and that Doležal's theorem will continue to hold if and only if both of these vector bundles are trivial. In particular, this result generalizes Doležal's theorem to the case where the parameter space is contractible (but possibly infinite-dimensional when $k = 0$) and subsumes the previously known results of Doležal et al.

© 2004 Elsevier Inc. All rights reserved.

Keywords: Doležal's theorem; Matrices depending on a parameter; Vector bundles; Contractible spaces; Block singular value decomposition

E-mail address: kgrasse@ou.edu

0024-3795/\$ - see front matter © 2004 Elsevier Inc. All rights reserved.

doi:10.1016/j.laa.2004.05.014

1. Introduction

This paper is concerned with matrices depending on a parameter and conditions under which certain standard types of matrix decompositions can be obtained that depend as “regularly” on the parameter as the original matrix. Considerable work has been done on this problem for the case of a scalar parameter in \mathbb{R} or \mathbb{C} (see, for example, [2,3,4,7,13,14,15,17]; these results will be briefly summarized as this introductory discussion unfolds), but noticeably less attention has been given to more general parameter spaces such as \mathbb{R}^n or topological spaces (see, for example, [6,10]). Here our principal objective is to extend what is known about certain parametric matrix decompositions from scalar to more general parameter spaces. We will focus on a specific type of decomposition embodied in a classical result of Doležal, which we will state after we introduce some elementary notation.

Notation 1.1. Following [2] we use \mathbb{F} to denote either the real numbers \mathbb{R} or the complex numbers \mathbb{C} . Given positive integers n and m we let $\mathbb{F}^{n \times m}$ denote the set of all $n \times m$ matrices with entries in \mathbb{F} . The complex-conjugate transpose of a matrix $A \in \mathbb{C}^{n \times m}$ is denoted by $A^* \in \mathbb{C}^{m \times n}$; however, when the entries of A are real, we write A^T in place of A^* . We use $GL(n, \mathbb{F})$ to stand for the group of all invertible $n \times n$ matrices with entries in \mathbb{F} . The *unitary* subgroup of $GL(n, \mathbb{C})$ will be denoted by $U(n, \mathbb{C})$ (so $A \in U(n, \mathbb{C}) \Leftrightarrow A^* = A^{-1}$), while the *orthogonal* subgroup of $GL(n, \mathbb{R})$ will be denoted by $O(n, \mathbb{R})$ (so $A \in O(n, \mathbb{R}) \Leftrightarrow A^T = A^{-1}$). By the neutral notation $U(n, \mathbb{F})$ we shall mean $U(n, \mathbb{C})$ if $\mathbb{F} = \mathbb{C}$ or $O(n, \mathbb{R})$ if $\mathbb{F} = \mathbb{R}$. We let I_n denote the $n \times n$ identity matrix and $0_{n \times m}$ denote the $n \times m$ matrix all of whose entries are zero. The dimensional subscripts from the zero matrix or identity matrix may be omitted if the dimensions can be contextually inferred. The symbol \mathbb{F}^0 will stand for the zero-dimensional (singleton) vector space $\mathbb{F}^0 = \{0\}$.

Theorem 1.2 (Doležal’s theorem [4]). *Let $J \subseteq \mathbb{R}$ be an interval, let $A : J \rightarrow \mathbb{F}^{n \times n}$ be a C^k mapping, $k = 0, 1, \dots, \infty$, and suppose that there exists an integer $r \in \{0, 1, \dots, n\}$ such that $\text{rank } A(t) = r$ for every $t \in J$. Then there exist C^k mappings $M : J \rightarrow GL(n, \mathbb{F})$ and $B : J \rightarrow \mathbb{F}^{n \times r}$ such that for every $t \in J$ we have*

$$A(t)M(t) = [B(t) \mid 0_{n \times (n-r)}]. \quad (1)$$

Remark 1.3. It is evident that the $n \times r$ matrix $B(t)$ must have rank r for every $t \in J$, so one concludes that for each $t \in J$ the r columns of $B(t)$ form a basis for the range of $A(t)$, while the last $n - r$ columns of $M(t)$ form a basis for the kernel of $A(t)$. The cases $r = 0$ and $r = n$ are, of course, trivial, but are included for completeness.

Shortly after the publication of Doležal’s paper [4], the closely related paper [14] of Sibuya appeared (which to the best of our knowledge was written independently

of, and more-or-less simultaneously with, [4]). The following analog of Theorem 1.2 is an immediate consequence of the main result of Sibuya's paper (see [14, Theorem 1]).

Theorem 1.4. *Let $D \subseteq \mathbb{C}$ be a simply connected domain, let $A : D \rightarrow \mathbb{C}^{n \times n}$ be a complex-analytic mapping, and suppose there exists an integer $r \in \{0, 1, \dots, n\}$ such that $\text{rank } A(z) = r$ for every $z \in D$. Then there exist complex analytic mappings $M : D \rightarrow GL(n, \mathbb{C})$ and $B : D \rightarrow \mathbb{C}^{n \times r}$ such that for every $z \in D$ we have*

$$A(z)M(z) = \begin{bmatrix} B(z) \\ 0_{n \times (n-r)} \end{bmatrix}.$$

Sibuya also obtains this result for real-valued matrix functions that are either real-analytic (that is, C^ω) or C^k ($k = 0, 1, \dots, \infty$) functions of a real variable $t \in \mathbb{R}$, so his results in fact subsume Doležal's theorem. Moreover, Sibuya's results treat periodic dependence on $t \in \mathbb{R}$ and allow one to infer corresponding periodicity properties of the matrix function $M(t)$ (see Theorems 3, 5, 6 and Remark 2 of [14]).

Other extensions and refinements of Doležal's theorem (for real scalar parameters) can be found in [13,15,17]. Interestingly, these papers do not cite Sibuya's paper [14], while at the same time Sibuya's paper seems to have generated a distinct thread of literature dealing with the general problem of parametric dependence of certain types of standard matrix decompositions, such as triangular, polar, QR , and block singular value; see, for example, [2,3,6,7].

The proofs of Theorem 1.2 in [4], Theorem 1.4 in [14], and the results of the consequent references [2,3,7,13,15,17] all appear to make essential use of the fact that the pertinent parameter space \mathbb{R} or \mathbb{C} is one-dimensional. Analogous results concerning more general parameter spaces are less prevalent. The paper [10] of Hsieh and Sibuya deals with matrices depending on parameters in the standard n -cube

$$\Omega = \{x = (x_1, \dots, x_n) \in \mathbb{F}^n \mid |x_j| \leq 1 \text{ for each } j = 1, \dots, n\} \quad (2)$$

and extends some of the results of Sibuya's paper [14] to this case. In particular, the results of [10] imply the analog of Theorem 1.4 when the simply connected region $D \subseteq \mathbb{C}$ is replaced by the complex n -cube $\Omega \subseteq \mathbb{C}^n$, and the analog of Theorem 1.2 when the interval J is replaced by the real n -cube $\Omega \subseteq \mathbb{R}^n$. However, the proof of the key result in [10] is carried out by induction on the dimension n of the ambient space and relies on the compactness of the n -cube Ω , so it does not appear to generalize in a straightforward manner to more general parameter spaces such as infinite-dimensional normed spaces. The paper of Gingold [6] deals with the somewhat related problem of continuous triangularization of matrices that depend on a parameter in a rectangular subset of \mathbb{R}^n , but here again inductive arguments are carried out on the dimension of the parameter space and thus do not seem to extend easily to more general parameter spaces.

Our objective in this paper is to use a vector-bundle approach to obtain an extension of Theorem 1.2 which, in the C^0 case, allows the parameter interval J to be replaced by any contractible topological space \mathcal{X} (recall that a topological space

\mathcal{X} is *contractible* if the identity mapping $1_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{X}$ is homotopic to a constant mapping). We will also derive differentiable versions of this result where the parameter space \mathcal{X} is either a contractible, finite-dimensional, differentiable manifold of class C^k for $k = 1, \dots, \infty, \omega$ or (when $\mathbb{F} = \mathbb{C}$) a contractible, finite-dimensional, complex-analytic Stein manifold (we will remind the reader of the definition of a Stein manifold in Section 3). As a consequence, our results will include the known generalizations of Doležal's theorem obtained from [10,14] as discussed above, but can also be applied in cases where the parameter space is infinite-dimensional. We are obliged to acknowledge that the possibility of using a vector-bundle approach is alluded to in [10], but the authors opted for a more “constructive” approach in their proof. The next example shows the necessity of placing some topological restrictions on the parameter space when seeking extensions of Doležal's theorem.

Example 1.5. Let \mathcal{X} denote the 2-sphere $S^2 \subseteq \mathbb{R}^3$ defined by

$$S^2 = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1 \right\},$$

and define a mapping $A : S^2 \rightarrow \mathbb{R}^{1 \times 3}$ by $A(x) = [x_1, x_2, x_3]$. Observe that A is the continuous (even real analytic) outer unit-normal vector field on S^2 , so in particular $\text{rank } A(x) = 1$ for every $x \in S^2$. We claim that there exists no continuous mapping $M : S^2 \rightarrow GL(3, \mathbb{R})$ such that

$$x \in S^2 \Rightarrow A(x)M(x) = [b(x), 0, 0], \quad (3)$$

where $b(x)$ is some (nonzero) scalar-valued function of $x \in S^2$. Indeed, if such a matrix function $M(x)$ were to exist, then (3) and the definition of $A(x)$ would imply immediately that either the second or third columns of the 3×3 matrix $M(x)$ would define a nowhere-zero, tangent vector field on S^2 . As is well known (see, for example, [5, p. 343] or [8, p. 134]), S^2 does not admit a continuous, nowhere-zero tangent vector field, which precludes the existence of a continuous mapping $M : S^2 \rightarrow GL(3, \mathbb{R})$ satisfying (3).

As we will see, the case of a contractible parameter space presents one simple, but still useful, situation in which topological obstructions to Doležal's theorem disappear. The previous example also hints that the situation in Doležal's theorem has some relationship to the theory of vector bundles. This relationship will be developed in Section 2. Our generalization of Doležal's theorem will be stated and proved in Section 3.

2. Parameterized matrix mappings and their associated vector bundles

This section contains a few basic definitions and reviews some elementary facts about vector bundles. We also associate to each constant-rank matrix-valued map-

ping $A : \mathcal{X} \rightarrow \mathbb{F}^{n \times m}$ a pair of vector bundles that will be used in our main theorem in Section 3.

Definition 2.1. Let s be a nonnegative integer. A *continuous (or C^0) s -plane vector bundle over the scalar field \mathbb{F}* consists of a pair of topological spaces \mathcal{E}, \mathcal{X} and a continuous surjection $\rho : \mathcal{E} \rightarrow \mathcal{X}$ with the property that for each $x \in \mathcal{X}$ there exist an open neighborhood \mathcal{U} of x and a homeomorphism $\tau_{\mathcal{U}} : \mathcal{U} \times \mathbb{F}^s \rightarrow \rho^{-1}(\mathcal{U})$ such that:

- (a) for every $(x, v) \in \mathcal{U} \times \mathbb{F}^s$ we have $(\rho \circ \tau_{\mathcal{U}})(x, v) = x$ (we call \mathcal{U} a *trivializing neighborhood* and $\tau_{\mathcal{U}}$ a *vector-bundle chart*);
- (b) for every pair of trivializing neighborhoods $\mathcal{U}, \mathcal{V} \subseteq \mathcal{X}$ with $\mathcal{U} \cap \mathcal{V} \neq \emptyset$ there exists a continuous mapping $\lambda_{\mathcal{U}, \mathcal{V}} : \mathcal{U} \cap \mathcal{V} \rightarrow GL(s, \mathbb{F})$ for which

$$\left((\tau_{\mathcal{V}})^{-1} \circ \tau_{\mathcal{U}} \right) (x, v) = (x, \lambda_{\mathcal{U}, \mathcal{V}}(x)v) \quad \forall (x, v) \in (\mathcal{U} \cap \mathcal{V}) \times \mathbb{F}^s.$$

Terminology 2.2. Given an s -plane vector bundle $\rho : \mathcal{E} \rightarrow \mathcal{X}$, we call \mathcal{E} the *total space*, \mathcal{X} the *base space*, and ρ the *projection mapping*. The vector-bundle charts induce on each fiber $\rho^{-1}(x) \stackrel{\text{def}}{=} \mathcal{E}_x$ a well defined s -dimensional vector space structure over the scalar field \mathbb{F} . We usually refer to the vector bundle $\rho : \mathcal{E} \rightarrow \mathcal{X}$ as *real* if $\mathbb{F} = \mathbb{R}$ and *complex* if $\mathbb{F} = \mathbb{C}$. If \mathcal{E} and \mathcal{X} are real finite-dimensional differentiable manifolds of class C^k for $k = 1, 2, \dots, \infty, \omega$ and if we replace “continuous” by “ C^k ” and “homeomorphism” by “ C^k diffeomorphism” in Definition 2.1, then we obtain the definition of a C^k s -plane vector bundle. Observe that even when $\mathbb{F} = \mathbb{C}$ the notion of a C^k vector bundle is an inherently “real” notion in the sense that we require the vector-bundle charts $\tau_{\mathcal{U}} : \mathcal{U} \times \mathbb{C}^s \rightarrow \rho^{-1}(\mathcal{U})$ to be of class C^k when \mathbb{C}^s is viewed as a real $2s$ -dimensional vector space (with an analogous statement for the group transition mappings $\lambda_{\mathcal{U}, \mathcal{V}}$). Alternatively, one can formulate a genuinely “complex” notion of vector bundle as follows. If $\mathbb{F} = \mathbb{C}$, if \mathcal{E} and \mathcal{X} are finite-dimensional, complex-analytic manifolds, and if we replace “continuous” by “complex analytic” and “homeomorphism” by “complex-analytic diffeomorphism” in Definition 2.1, then we obtain the definition of a complex-analytic, or holomorphic, s -plane vector bundle. Consequently, we make a subtle distinction between complex vector bundles that are C^k and complex vector bundles that are holomorphic. See [8, Chapter 4], [12, Chapter 2], or [18, Chapter 1] for more details.

The next proposition establishes a natural connection between vector bundles and parameterized matrix mappings.

Proposition 2.3. Let \mathcal{X} be a Hausdorff topological space and let $A : \mathcal{X} \rightarrow \mathbb{F}^{n \times m}$ be a continuous $n \times m$ -matrix-valued mapping with the property that for some integer $r = 0, \dots, \min\{n, m\}$ we have $\text{rank}_{\mathbb{F}} A(x) = r$ for every $x \in \mathcal{X}$. Then the set

$$\mathcal{R}_A = \{(x, v) \in \mathcal{X} \times \mathbb{F}^n \mid \exists u \in \mathbb{F}^m \text{ such that } v = A(x)u\},$$

when topologized as a subspace of $\mathcal{X} \times \mathbb{F}^n$ and equipped with the projection mapping $\rho : \mathcal{R}_A \rightarrow \mathcal{X}$, $(x, v) \mapsto x$, has the structure of a C^0 r -plane vector bundle over \mathcal{X} . Moreover, the set

$$\mathcal{K}_A = \{(x, u) \in \mathcal{X} \times \mathbb{F}^m \mid A(x)u = 0\},$$

when topologized as a subspace of $\mathcal{X} \times \mathbb{F}^m$ and equipped with the projection mapping $\rho' : \mathcal{K}_A \rightarrow \mathcal{X}$, $(x, u) \mapsto x$, has the structure of a $C^0(m - r)$ -plane vector bundle over \mathcal{X} .

Proof. We will carry out the argument under the assumption that $1 \leq r < \min\{n, m\}$. The extreme cases $r = 0$ and $r = \min\{n, m\}$ can be handled by a straightforward modification of the proof given below (note, however, that one of the fiber dimensions will degenerate to 0 if either $r = 0$ or $r = m \leq n$). Our proof hinges on the following claim. The proof is omitted since it is a fairly straightforward consequence of the constant-rank assumption imposed on the matrix function $x \mapsto A(x)$ (see [18, pp. 6–7] for the essence of the argument).

Claim. For each $x_0 \in \mathcal{X}$ there exist an open neighborhood \mathcal{U} of x_0 and C^0 matrix functions $P : \mathcal{U} \rightarrow GL(n, \mathbb{F})$, $Q : \mathcal{U} \rightarrow GL(m, \mathbb{F})$ such that

$$x \in \mathcal{U} \Rightarrow P(x)A(x)Q(x) = \begin{bmatrix} I_r & 0_{r \times (m-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (m-r)} \end{bmatrix}. \quad (4)$$

Using the claim we can easily establish the desired vector bundle structure on the spaces \mathcal{R}_A and \mathcal{K}_A . Let $x_0 \in \mathcal{X}$ and let $P : \mathcal{U} \rightarrow GL(n, \mathbb{F})$, $Q : \mathcal{U} \rightarrow GL(m, \mathbb{F})$ be C^k mappings defined in an open neighborhood \mathcal{U} of x_0 such that (4) holds for every $x \in \mathcal{U}$. For $x \in \mathcal{U}$ and for (column) vectors $v \in \mathbb{F}^n$ and $u \in \mathbb{F}^m$ we have

$$\begin{aligned} v = A(x)u &\Leftrightarrow P(x)v = P(x)A(x)Q(x)Q(x)^{-1}u \\ &\Leftrightarrow P(x)v = \begin{bmatrix} I_r & 0_{r \times (m-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (m-r)} \end{bmatrix} Q(x)^{-1}u \\ &\Leftrightarrow v = P(x)^{-1} \begin{bmatrix} w \\ 0 \end{bmatrix}, \end{aligned}$$

where $w \in \mathbb{F}^r$ and $0 \in \mathbb{F}^{n-r}$ are the column vectors defined by the relation

$$\begin{bmatrix} w \\ 0 \end{bmatrix} = \begin{bmatrix} I_r & 0_{r \times (m-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (m-r)} \end{bmatrix} Q(x)^{-1}u.$$

Thus the mapping of $\mathcal{U} \times \mathbb{F}^r$ onto $\rho^{-1}(\mathcal{U})$ given by

$$(x, w) \mapsto \left(x, P(x)^{-1} \begin{bmatrix} w \\ 0 \end{bmatrix} \right)$$

is a homeomorphism and it is evident that any two such maps will overlap according to Definition 2.1(b). Consequently $\rho : \mathcal{R}_A \rightarrow \mathcal{X}$ has the structure of a C^0 vector bundle as claimed.

Turning our attention to \mathcal{K}_A , we note that for $x \in \mathcal{U}$ and a (column) vector $u \in \mathbb{F}^m$ we have

$$\begin{aligned} A(x)u = 0 &\Leftrightarrow P(x)A(x)Q(x)Q(x)^{-1}u = 0 \\ &\Leftrightarrow \begin{bmatrix} I_r & 0_{r \times (m-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (m-r)} \end{bmatrix} Q(x)^{-1}u = 0. \end{aligned}$$

This forces $Q(x)^{-1}u$ to have the form

$$Q(x)^{-1}u = \begin{bmatrix} 0 \\ z \end{bmatrix} \Leftrightarrow u = Q(x) \begin{bmatrix} 0 \\ z \end{bmatrix},$$

where $0 \in \mathbb{F}^r$ and $z \in \mathbb{F}^{m-r}$. Thus the mapping of $\mathcal{U} \times \mathbb{F}^{m-r}$ onto $(\rho')^{-1}(\mathcal{U})$ given by

$$(x, z) \mapsto \left(x, Q(x) \begin{bmatrix} 0 \\ z \end{bmatrix} \right)$$

is a homeomorphism and it is evident that any two such maps will overlap according to Definition 2.1(b). Consequently $\rho' : \mathcal{K}_A \rightarrow \mathcal{X}$ has the structure of a C^0 vector bundle as claimed and the proof is complete. \square

Corollary 2.4. *Let $A : \mathcal{X} \rightarrow \mathbb{F}^{n \times m}$ be as in the statement of the theorem.*

- (a) *If \mathcal{X} also has the structure of a real, finite-dimensional, differentiable manifold of class C^k for $k = 1, \dots, \infty, \omega$ and if A is a C^k mapping, then the vector bundles $\rho : \mathcal{R}_A \rightarrow \mathcal{X}$ and $\rho' : \mathcal{K}_A \rightarrow \mathcal{X}$ are C^k .*
- (b) *If $\mathbb{F} = \mathbb{C}$, if \mathcal{X} also has the structure of a finite-dimensional, complex-analytic manifold, and if A is a complex-analytic mapping, then the vector bundles $\rho : \mathcal{R}_A \rightarrow \mathcal{X}$ and $\rho' : \mathcal{K}_A \rightarrow \mathcal{X}$ are holomorphic.*

Proof. The proofs of these assertions are almost identical to the proof given in the continuous case because all matrix functions occurring in that proof are easily seen to be C^k (resp., complex analytic) if A is C^k (resp., complex analytic). \square

Remark 2.5. Recall that for $k = 0, 1, \dots, \infty, \omega$ a C^k s -plane bundle $\rho : \mathcal{E} \rightarrow \mathcal{X}$ is C^k *trivial* if it is C^k vector-bundle isomorphic to the product bundle $\pi : \mathcal{X} \times \mathbb{F}^s \rightarrow \mathcal{X}$, where π denotes the projection on the first factor (so there exists a C^k diffeomorphism $\Lambda : \mathcal{E} \rightarrow \mathcal{X} \times \mathbb{F}^s$ such that $\pi \circ \Lambda = \rho$ and Λ is a linear isomorphism on each fiber). There is a similarly formulated notion of triviality for a holomorphic vector bundle. Also recall that a *section* of a vector bundle $\rho : \mathcal{E} \rightarrow \mathcal{X}$ is a mapping $\sigma : \mathcal{X} \rightarrow \mathcal{E}$ such that $\rho \circ \sigma = 1_{\mathcal{X}}$ (the identity mapping on \mathcal{X}). If $\rho : \mathcal{E} \rightarrow \mathcal{X}$ is a C^k vector bundle and $\ell = 0, 1, \dots, \infty, \omega$ is such that $\ell \leq k$, then we let

$\Gamma^\ell(\mathcal{X}, \mathcal{E})$ denote the set of all C^ℓ sections of \mathcal{E} . Similarly, the set of all complex-analytic sections of a holomorphic vector bundle $\rho : \mathcal{E} \rightarrow \mathcal{X}$ will be denoted by $\Gamma^{ca}(\mathcal{X}, \mathcal{E})$.

Proposition 2.6. *Let $k = 0, 1, \dots, \infty, \omega$. A C^k (resp., holomorphic)s-plane bundle $\rho : \mathcal{E} \rightarrow \mathcal{X}$ is C^k (resp., holomorphically) trivial if and only if there exist s sections $\sigma_1, \dots, \sigma_s$ in $\Gamma^k(\mathcal{X}, \mathcal{E})$ (resp., in $\Gamma^{ca}(\mathcal{X}, \mathcal{E})$) such that for each $x \in \mathcal{X}$ the vectors $\sigma_1(x), \dots, \sigma_s(x)$ are linearly independent over \mathbb{F} in the fiber \mathcal{E}_x .*

Proof. See [12, p. 18]. \square

3. The main theorem

In this section we prove our main result, which states that a generalized version of Doležal's theorem holds for a parameterized matrix mapping $A : \mathcal{X} \rightarrow \mathbb{F}^{n \times m}$ if and only if both of the vector bundles $\rho : \mathcal{R}_A \rightarrow \mathcal{X}$ and $\rho' : \mathcal{K}_A \rightarrow \mathcal{X}$ defined in Proposition 2.3 are trivial. We will then appeal to a standard result from the theory of vector bundles to show that these bundles are indeed C^0 trivial if the base space \mathcal{X} is contractible. Although not entirely obvious, it turns out that a C^k vector bundle over a contractible space is also C^k trivial for $k = 1, \dots, \infty, \omega$, a fact which is essential for our purposes. In the complex-analytic case, however, it appears that one must assume the base space \mathcal{X} is a contractible Stein manifold to infer that a holomorphic vector bundle $\rho : \mathcal{E} \rightarrow \mathcal{X}$ is holomorphically trivial. We will appeal to appropriate references as we sort through these technical issues.

First we start with a basic fact from linear algebra whose straightforward proof is omitted.

Lemma 3.1. *Let $A \in \mathbb{F}^{n \times m}$ be of rank r over \mathbb{F} with $1 \leq r < m$, let V be an $n \times r$ matrix whose columns are linearly independent and span the range of A , and let U be an $m \times (m - r)$ matrix whose columns are linearly independent and span the kernel of A . Then there exists a unique $m \times r$ matrix W such that*

$$V^*AW = V^*V \quad \text{and} \quad U^*W = 0_{(m-r) \times r}. \quad (5)$$

Moreover, the matrix W satisfying (5) is given by the formula

$$W = A^*V(V^*AA^*V)^{-1}V^*V,$$

and satisfies $AW = V$.

Theorem 3.2. *Let \mathcal{X} be a Hausdorff topological space and let $A : \mathcal{X} \rightarrow \mathbb{F}^{n \times m}$ be a continuous $n \times m$ -matrix-valued mapping with the property that for some integer $r = 0, \dots, \min\{n, m\}$ we have $\text{rank}_{\mathbb{F}} A(x) = r$ for every $x \in \mathcal{X}$. Then the following statements are equivalent.*

(a) There exist C^0 mappings $M : \mathcal{X} \rightarrow U(m, \mathbb{F})$ and $B : \mathcal{X} \rightarrow \mathbb{F}^{n \times r}$ such that

$$x \in \mathcal{X} \Rightarrow A(x)M(x) = \begin{bmatrix} B(x) \\ 0_{n \times (m-r)} \end{bmatrix}. \quad (6)$$

(b) There exist C^0 mappings $M : \mathcal{X} \rightarrow GL(m, \mathbb{F})$ and $B : \mathcal{X} \rightarrow \mathbb{R}^{n \times r}$ such that Eq. (6) holds.

(c) Both of the vector bundles $\rho : \mathcal{R}_A \rightarrow \mathcal{X}$ and $\rho' : \mathcal{K}_A \rightarrow \mathcal{X}$ defined in Proposition 2.3 are C^0 trivial.

Proof. The extreme cases where the matrix rank satisfies $r = 0$ or $r = m \leq n$ are easily handled, so we will carry out the argument under the assumption that $0 < r < m$.

(a) \Rightarrow (b): this is obvious.

(b) \Rightarrow (c): let $M : \mathcal{X} \rightarrow GL(m, \mathbb{F})$ be a C^0 mapping satisfying (6) and for $x \in \mathcal{X}$ and $i = 1, \dots, m$ let $\mu_i(x)$ denote the i th column of the matrix $M(x)$. It is clear that $A(x)\mu_i(x) = 0$ for each $i = r+1, \dots, m$ and $x \in \mathcal{X}$, so it follows that for $i = r+1, \dots, m$ the mappings $\sigma_i : \mathcal{X} \rightarrow \mathcal{X} \times \mathbb{F}^m$, $\sigma_i(x) = (x, \mu_i(x))$ form a set of $m-r$ linearly independent, C^0 sections of the $(m-r)$ -plane vector bundle \mathcal{K}_A , whence \mathcal{K}_A is trivial by Proposition 2.6.

Next observe that $\text{rank}_{\mathbb{F}} A(x) = \text{rank}_{\mathbb{F}} A(x)M(x) = r$ for every $x \in \mathcal{X}$, because $M(x)$ is invertible. Since $A(x)\mu_i(x) = 0$ for $i = r+1, \dots, m$, we infer that the column vectors $A(x)\mu_1(x), \dots, A(x)\mu_r(x)$ are linearly independent for every $x \in \mathcal{X}$. Consequently, for $i = 1, \dots, r$ the mappings $\sigma_i : \mathcal{X} \rightarrow \mathcal{X} \times \mathbb{F}^n$, $\sigma_i(x) = (x, A(x)\mu_i(x))$ form a set of r linearly independent, C^0 sections of the r -plane bundle \mathcal{R}_A , whence by Proposition 2.6 \mathcal{R}_A is trivial as well.

(c) \Rightarrow (a): The assumption that \mathcal{K}_A is C^0 trivial implies that there exist $m-r$ linearly independent, C^0 sections $\sigma_{r+1}, \dots, \sigma_m : \mathcal{X} \rightarrow \mathcal{K}_A$ of the $m-r$ -plane bundle \mathcal{K}_A . Let $\pi_2 : \mathcal{K}_A \rightarrow \mathbb{F}^m$ denote the restriction of the projection mapping $\mathcal{X} \times \mathbb{F}^m \rightarrow \mathbb{F}^m$ to \mathcal{K}_A , and set $u_i = \pi_2 \circ \sigma_i$ for $i = r+1, \dots, m$. Evidently, the mappings $u_{r+1}, \dots, u_m : \mathcal{X} \rightarrow \mathbb{F}^m$ are C^0 and for every $x \in \mathcal{X}$ the vectors $u_{r+1}(x), \dots, u_m(x)$ are linearly independent and span the kernel of $A(x)$.

Likewise, the assumption that \mathcal{R}_A is trivial implies that there exist r linearly independent, C^0 sections $\sigma_1, \dots, \sigma_r : \mathcal{X} \rightarrow \mathcal{R}_A$ of the r -plane bundle \mathcal{R}_A . Let $\pi'_2 : \mathcal{R}_A \rightarrow \mathbb{F}^n$ denote the restriction of the projection mapping $\mathcal{X} \times \mathbb{F}^n \rightarrow \mathbb{F}^n$ to \mathcal{R}_A , and set $v_i = \pi'_2 \circ \sigma_i$ for $i = 1, \dots, r$. Evidently, the mappings $v_1, \dots, v_r : \mathcal{X} \rightarrow \mathbb{F}^n$ are C^0 and for every $x \in \mathcal{X}$ the vectors $v_1(x), \dots, v_r(x)$ are linearly independent and span the range of $A(x)$.

Now define the C^0 matrix-valued mappings $U : \mathcal{X} \rightarrow \mathbb{F}^{m \times (m-r)}$ and $V : \mathcal{X} \rightarrow \mathbb{F}^{n \times r}$ by

$$U(x) = [u_{r+1}(x), \dots, u_m(x)], \quad V(x) = [v_1(x), \dots, v_r(x)].$$

Since for each $x \in \mathcal{X}$ the columns of $U(x)$ span the kernel of $A(x)$ and the columns of $V(x)$ span the range of $A(x)$, Lemma 3.1 implies that for each $x \in \mathcal{X}$ the $m \times r$ matrix

$$W(x) = A(x)^* V(x) (V(x)^* A(x) A(x)^* V(x))^{-1} V(x)^* V(x) \quad (7)$$

satisfies $A(x)W(x) = V(x)$ and $U(x)^*W(x) = 0$. It is evident from (7) that the resulting matrix mapping $W : \mathcal{X} \rightarrow \mathbb{R}^{m \times r}$ is C^0 because the same is true of the matrix functions A and V . For each $x \in \mathcal{X}$ let $\tilde{U}(x)$ denote the $m \times (m-r)$ obtained from $U(x)$ by applying the Gram–Schmidt orthonormalization process sequentially (from left to right) to the $m-r$ columns of $U(x)$. Likewise for each $x \in \mathcal{X}$ let $\tilde{W}(x)$ denote the $m \times r$ matrix obtained from $W(x)$ by applying the Gram–Schmidt orthonormalization process sequentially to the r columns of $W(x)$. Observe that the resulting mappings

$$\tilde{U} : \mathcal{X} \rightarrow \mathbb{F}^{m \times (m-r)}, \quad \tilde{W} : \mathcal{X} \rightarrow \mathbb{F}^{m \times r},$$

are C^0 because the same is true of the mappings U and W , and the vectors produced by the Gram–Schmidt process depend continuously on the vectors to which the process is applied. Also observe that we have $\tilde{U}(x)^*\tilde{W}(x) = 0$ for each $x \in \mathcal{X}$ since we have $U(x)^*W(x) = 0$ and the column spans of $\tilde{U}(x)$ and $\tilde{W}(x)$ coincide with the column spans of $U(x)$ and $W(x)$, respectively. Thus for each $x \in \mathcal{X}$ the columns of the $m \times m$ matrix $M(x) = [\tilde{W}(x), \tilde{U}(x)]$ form an orthonormal basis of \mathbb{F}^m , so it is now apparent that the resulting matrix mapping $M : \mathcal{X} \rightarrow U(m, \mathbb{F})$ satisfies our requirements. \square

Corollary 3.3. *Let \mathcal{X} and A be as in the statement of the theorem. If \mathcal{X} also has the structure of a real, finite-dimensional, differentiable manifold of class C^k for $k = 1, \dots, \infty, \omega$ and if A is a C^k mapping, then the statements (a), (b), and (c) of the theorem are equivalent if we replace C^0 by C^k .*

Proof. One simply follows the proof of the Theorem and replaces C^0 by C^k . One perhaps should observe that the entries of the matrix $W(x)$ (or, if $\mathbb{F} = \mathbb{C}$, the real and imaginary parts of the entries of $W(x)$) given by (7) are C^k functions of x whenever A and V are C^k functions of x , since the operations of matrix inverse and (complex-conjugate) transpose are real analytic. A similar remark applies to the vectors produced by the Gram–Schmidt process on a family of C^k vector functions. \square

Unfortunately, the proof of Theorem 3.2 does not carry over directly to the complex-analytic case because the operation of complex-conjugate transpose and the Gram–Schmidt process are not complex analytic due to the presence of the conjugation operation. To obtain a version of Theorem 3.2 in the complex-analytic case one has to use some technical machinery from the theory of functions of several complex variables and put additional restrictions on the base manifold \mathcal{X} to ensure that holomorphic vector bundles over \mathcal{X} have an “ample” number of global holomorphic sections. Specifically, we call a complex analytic manifold \mathcal{X} a *Stein manifold* if \mathcal{X} can be holomorphically embedded as a closed submanifold of \mathbb{C}^N for some $N \in \mathbb{N}$ sufficiently large. We note that this is not the standard way of defining a Stein

manifold, but it is the most convenient for our purposes. For the standard definition, as well as the proof of its equivalence to the one we have provided, we refer the reader to the book of Hörmander ([9, Chapter 5]). Although not apparent from our definition, it can be shown that every convex open subset of \mathbb{C}^n is a Stein manifold (see [9]).

One key result concerning the theory of Stein manifolds to which we will appeal is a consequence of the so-called “Theorem A of Cartan” (see [9, p. 174]), which in essence asserts the existence of an ample number of holomorphic sections of coherent analytic sheaves over Stein manifolds. Any attempt to summarize the necessary terminology to present a statement of the sheaf-theoretic version of this result would take us too far afield from our principal objectives, so we will be content to state a special case that is adequate for our purposes (see [9, Theorem 7.2.9] for the proof).

Proposition 3.4. *Let \mathcal{X} be a Stein manifold and let $A : \mathcal{X} \rightarrow \mathbb{C}^{n \times m}$ and $v : \mathcal{X} \rightarrow \mathbb{C}^n$ be complex-analytic mappings with the following property: for each $x_0 \in \mathcal{X}$ there exist an open neighborhood \mathcal{U} of x_0 and complex-analytic functions $\gamma_1, \dots, \gamma_m : \mathcal{U} \rightarrow \mathbb{C}$ (possibly depending on x_0) such that*

$$x \in \mathcal{U} \Rightarrow v(x) = \sum_{j=1}^m \gamma_j(x) a_j(x),$$

where $a_j(x)$ denotes the j th column of the $n \times m$ matrix $A(x)$. Then there exist globally defined, complex-analytic functions $c_1, \dots, c_m : \mathcal{X} \rightarrow \mathbb{C}$ such that

$$x \in \mathcal{X} \Rightarrow v(x) = \sum_{j=1}^m c_j(x) a_j(x).$$

Theorem 3.5. *Let \mathcal{X} be a Stein manifold and let $A : \mathcal{X} \rightarrow \mathbb{C}^{n \times m}$ be a complex-analytic $n \times m$ -matrix-valued mapping with the property that for some integer $r = 0, \dots, \min\{n, m\}$ we have $\text{rank}_{\mathbb{C}} A(x) = r$ for every $x \in \mathcal{X}$. Then the following statements are equivalent.*

- (a) *There exist complex-analytic mappings $M : \mathcal{X} \rightarrow GL(m, \mathbb{C})$ and $B : \mathcal{X} \rightarrow \mathbb{C}^{n \times r}$ such that*

$$x \in \mathcal{X} \Rightarrow A(x)M(x) = [B(x) \mid 0_{n \times (m-r)}]. \quad (8)$$

- (b) *Both of the vector bundles $\rho : \mathcal{R}_A \rightarrow \mathcal{X}$ and $\rho' : \mathcal{K}_A \rightarrow \mathcal{X}$ defined in Proposition 2.3 are holomorphically trivial.*

Proof. (a) \Rightarrow (b): This follows exactly as in the proof of the implication (b) \Rightarrow (c) in Theorem 3.2 if we simply set $\mathbb{F} = \mathbb{C}$ and replace “ C^0 ” by “complex analytic”.

(b) \Rightarrow (a): The assumption that \mathcal{K}_A is holomorphically trivial implies by Proposition 2.6 that there exist $m - r$ linearly independent, complex-analytic sections

$\sigma_{r+1}, \dots, \sigma_m : \mathcal{X} \rightarrow \mathcal{K}_A$ of the $m - r$ -plane bundle \mathcal{K}_A . Let $\pi_2 : \mathcal{K}_A \rightarrow \mathbb{C}^m$ denote the restriction of the projection mapping $\mathcal{X} \times \mathbb{C}^m \rightarrow \mathbb{C}^m$ to \mathcal{K}_A , and set $u_k = \pi_2 \circ \sigma_k$ for $k = r + 1, \dots, m$. Evidently, the mappings $u_{r+1}, \dots, u_m : \mathcal{X} \rightarrow \mathbb{C}^m$ are complex analytic and for every $x \in \mathcal{X}$ the vectors $u_{r+1}(x), \dots, u_m(x)$ are linearly independent and span the kernel of $A(x)$.

Likewise, the assumption that \mathcal{R}_A is holomorphically trivial implies that there exist r linearly independent, complex-analytic sections $\sigma_1, \dots, \sigma_r : \mathcal{X} \rightarrow \mathcal{R}_A$ of the r -plane bundle \mathcal{R}_A . Let $\pi'_2 : \mathcal{R}_A \rightarrow \mathbb{C}^n$ denote the restriction of the projection mapping $\mathcal{X} \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ to \mathcal{R}_A , and set $v_k = \pi'_2 \circ \sigma_k$ for $k = 1, \dots, r$. Evidently, the mappings $v_1, \dots, v_r : \mathcal{X} \rightarrow \mathbb{C}^n$ are complex analytic and for every $x \in \mathcal{X}$ the vectors $v_1(x), \dots, v_r(x)$ are linearly independent and span the range of $A(x)$.

We claim that for each $k = 1, \dots, r$ there exists a complex-analytic mapping $w_k : \mathcal{X} \rightarrow \mathbb{C}^m$ such that $A(x)w_k(x) = v_k(x)$ for every $x \in \mathcal{X}$. Indeed by the assumption that \mathcal{X} is a Stein manifold and Proposition 3.4 it is enough to prove that for each $k = 1, \dots, r$ and for every $x_0 \in \mathcal{X}$ there exist an open neighborhood \mathcal{U} of x_0 and a complex-analytic mapping $\gamma_k : \mathcal{U} \rightarrow \mathbb{C}^m$ such that $A(x)\gamma_k(x) = v_k(x)$ for every $x \in \mathcal{U}$. However, this is seen to be a direct consequence of the (holomorphic version of the) Claim in Proposition 2.3 due to the constant-rank assumption on A , so the desired mappings $w_k, k = 1, \dots, r$ are seen to exist as claimed.

We next define complex-analytic mappings $M : \mathcal{X} \rightarrow \mathbb{C}^{m \times m}$ and $B : \mathcal{X} \rightarrow \mathbb{C}^{n \times r}$ in terms of the complex-analytic, m -dimensional (column) vector functions $w_1, \dots, w_r, u_{r+1}, \dots, u_m : \mathcal{X} \rightarrow \mathbb{C}^m$, by

$$M(x) = [w_1(x), \dots, w_r(x), u_{r+1}(x), \dots, u_m(x)]$$

and

$$B(x) = A(x)[w_1(x), \dots, w_r(x)].$$

It is easy to see that $M(x)$ is invertible for each $x \in \mathcal{X}$ and M and B satisfy (8), so the proof is complete. \square

Remark 3.6. Observe that one would not expect the analog of assertion (a) of Theorem 3.2 to hold in the complex-analytic case since, for example, if we had $\mathcal{X} = \mathbb{C}$ and $M : \mathbb{C} \rightarrow U(n, \mathbb{C})$ complex analytic, then the compactness of $U(n, \mathbb{C})$ would force M to be a constant.

Theorem 3.2, Corollary 3.3, and Theorem 3.5 all assert that various versions of Doležal's theorem hold when the vector bundles $\rho : \mathcal{R}_A \rightarrow \mathcal{X}$ and $\rho' : \mathcal{K}_A \rightarrow \mathcal{X}$ are trivial. The next result recalls an important and useful situation in which the C^k (or holomorphic, if appropriate) triviality of a vector bundle $\rho : \mathcal{E} \rightarrow \mathcal{X}$ is assured.

Theorem 3.7

(a) *If \mathcal{X} is a contractible, paracompact, Hausdorff space, then every C^0 s -plane vector bundle $\rho : \mathcal{E} \rightarrow \mathcal{X}$ is C^0 trivial.*

- (b) If \mathcal{X} is a contractible, paracompact, Hausdorff, finite-dimensional differentiable manifold of class C^k for $k = 1, \dots, \infty, \omega$, then every C^k s -plane vector bundle $\rho : \mathcal{E} \rightarrow \mathcal{X}$ is C^k trivial.
- (c) If \mathcal{X} is a contractible Stein manifold, then every holomorphic s -plane vector bundle $\rho : \mathcal{E} \rightarrow \mathcal{X}$ is holomorphically trivial.

Proof. For the proofs of assertion (a) and assertion (b) in the cases where $k = 1, \dots, \infty$ see [8, p. 97]. The proof of (b) when $k = \omega$ and the proof of (c) are likely part of the folklore in the subject of vector bundles, but we were not able to locate any precise references, so we offer very brief sketches of the requisite arguments.

The proof of (b) in the C^ω case can be deduced from the C^∞ case via the following approximation argument. Since a C^ω vector bundle $\rho : \mathcal{E} \rightarrow \mathcal{X}$ is, in particular, a C^∞ vector bundle, the C^∞ version of (b) implies that the C^ω s -plane bundle $\rho : \mathcal{E} \rightarrow \mathcal{X}$ is C^∞ vector-bundle isomorphic to a trivial bundle, so there exist C^∞ sections $\sigma_1, \dots, \sigma_s \in \Gamma^\infty(\mathcal{X}, \mathcal{E})$ such that for each $x \in \mathcal{X}$ the vectors $\sigma_1(x), \dots, \sigma_s(x)$ are linearly independent in the fiber \mathcal{E}_x . A well known, but deep, result states that the space of real-analytic sections $\Gamma^\omega(\mathcal{X}, \mathcal{E})$ is a dense subset of $\Gamma^\infty(\mathcal{X}, \mathcal{E})$ in the fine C^∞ topology (see [11, p. 303]), and the nature of the fine C^∞ topology allows us to obtain for each $i = 1, \dots, s$ a real-analytic section $\tilde{\sigma}_i \in \Gamma^\omega(\mathcal{X}, \mathcal{E})$ such that for each $x \in \mathcal{X}$ the vectors $\tilde{\sigma}_1(x), \dots, \tilde{\sigma}_s(x)$ are linearly independent in the fiber \mathcal{E}_x , whence $\rho : \mathcal{E} \rightarrow \mathcal{X}$ is C^ω trivial by Proposition 2.6.

The proof of the (real) C^ω case cannot be used to prove assertion (c) because C^∞ sections of a holomorphic vector bundle cannot generally be approximated by complex-analytic sections. Thus we resort to yet another argument. Given a holomorphic s -plane vector bundle $\rho : \mathcal{E} \rightarrow \mathcal{X}$ over a contractible Stein manifold \mathcal{X} we form its associated *principal fiber bundle* $\tilde{\rho} : \mathcal{B} \rightarrow \mathcal{X}$ (see [16, pp. 35–36] for details). One knows that the holomorphic vector bundle $\rho : \mathcal{E} \rightarrow \mathcal{X}$ is holomorphically (resp., C^k) trivial if and only if its associated principal bundle $\tilde{\rho} : \mathcal{B} \rightarrow \mathcal{X}$ admits a global holomorphic (resp., global C^k) section. However, there will certainly exist a *continuous* section $\sigma : \mathcal{X} \rightarrow \mathcal{B}$ because $\tilde{\rho} : \mathcal{B} \rightarrow \mathcal{X}$ is, in particular, a C^0 fiber bundle over the contractible topological space \mathcal{X} , and thus is C^0 trivial by the analog of assertion (a) for topological fiber bundles (see [16, p. 53]). To obtain the existence of a global complex-analytic section we appeal to a result of Cartan (see [1, Theorem 1]), which states that every continuous section of a holomorphic principal fiber bundle with group $GL(s, \mathbb{C})$ over a Stein manifold is homotopic to a complex-analytic section. In particular, the principal fiber bundle $\tilde{\rho} : \mathcal{B} \rightarrow \mathcal{X}$ has a globally defined complex analytic section because it has a globally defined continuous section (being C^0 trivial because its base space \mathcal{X} is contractible). Thus $\rho : \mathcal{E} \rightarrow \mathcal{X}$ is holomorphically trivial as required. \square

Theorem 3.8 (Doležal's theorem for contractible parameter spaces). *Let \mathcal{X} be a topological space that is Hausdorff, paracompact, and contractible, and let $A : \mathcal{X} \rightarrow$*

$\mathbb{F}^{n \times m}$ be a continuous $n \times m$ -matrix-valued mapping with the property that for some integer $r = 0, \dots, \min\{n, m\}$ we have $\text{rank}_{\mathbb{F}} A(x) = r$ for every $x \in \mathcal{X}$.

(a) There exist C^0 mappings $M : \mathcal{X} \rightarrow U(m, \mathbb{F})$ and $B : \mathcal{X} \rightarrow \mathbb{F}^{n \times r}$ such that

$$x \in \mathcal{X} \Rightarrow A(x)M(x) = \begin{bmatrix} B(x) \\ 0_{n \times (m-r)} \end{bmatrix}. \quad (9)$$

(b) If \mathcal{X} has the structure of a real, finite-dimensional, differentiable manifold of class C^k for $k = 1, \dots, \infty, \omega$ and if A is of class C^k , then there exist C^k mappings $M : \mathcal{X} \rightarrow U(m, \mathbb{F})$ and $B : \mathcal{X} \rightarrow \mathbb{F}^{n \times r}$ such that (9) holds.

(c) If $\mathbb{F} = \mathbb{C}$, if \mathcal{X} has the structure of a complex-analytic Stein manifold, and if A is complex analytic, then there exist complex-analytic mappings $M : \mathcal{X} \rightarrow GL(m, \mathbb{C})$ and $B : \mathcal{X} \rightarrow \mathbb{C}^{n \times r}$ such that (9) holds.

Proof. The constant-rank assumption on the matrix-valued mapping A allows us to form the C^0 r -plane bundle $\rho : \mathcal{R}_A \rightarrow \mathcal{X}$ and the C^0 $(m - r)$ -plane bundle $\rho' : \mathcal{K}_A \rightarrow \mathcal{X}$ given by Proposition 2.3. Our assumptions imply that both of these bundles are C^0 trivial by Theorem 3.7(a), so assertion (a) is a consequence of Theorem 3.2. If \mathcal{X} is a C^k manifold and A is a C^k mapping, then the bundles $\rho : \mathcal{R}_A \rightarrow \mathcal{X}$ and $\rho' : \mathcal{K}_A \rightarrow \mathcal{X}$ are of class C^k and are C^k trivial by Theorem 3.7(b), so assertion (b) is a consequence of Corollary 3.3. Likewise, if \mathcal{X} is a complex-analytic Stein manifold and A is complex analytic, then these bundles are holomorphic and are holomorphically trivial by Theorem 3.7(c), so assertion (c) follows from Theorem 3.5. This completes the proof. \square

Remark 3.9. The extension of Doležal's theorem to parameter spaces that are contractible topological spaces (or contractible real or complex manifolds) is of interest for at least two reasons. First, it is known that there are contractible manifolds of dimension $n = 3$ that are not homeomorphic to the interior of the standard 3-cube $\Omega \subseteq \mathbb{R}^3$ (see [19]). Admittedly, such examples are somewhat esoteric and are not likely to impinge on applications, but they do show that an extension of Doležal's theorem from the case where the parameter space is the cube $\Omega \subseteq \mathbb{F}^n$ to a more general contractible space has some mathematical significance. Second, one can envisage practical situations where it is desirable to apply Doležal's theorem when the parameter space is contractible and infinite-dimensional, and such situations are not covered by the known results.

References

- [1] H. Cartan, Espaces fibrés analytiques, in: Symposium International de Topologia Algebraica, Mexico, 1958, pp. 97–121.
- [2] J.-L. Chern, L. Dieci, Smoothness and periodicity of some matrix decompositions, SIAM J. Matrix Anal. Appl. 22 (2000) 772–792.

- [3] L. Dieci, T. Eirola, On smooth decompositions of matrices, *SIAM J. Matrix Anal. Appl.* 20 (1999) 800–819.
- [4] V. Doležal, The existence of a continuous basis of a certain linear subspace of E_r which depends on a parameter, *Časopis Pěst. Mat.* 89 (1964) 466–469.
- [5] J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, MA, 1966.
- [6] H. Gingold, On continuous triangularization of matrix functions, *SIAM J. Math. Anal.* 10 (1979) 709–720.
- [7] H. Gingold, P.-F. Hsieh, Globally analytic triangularization of a matrix function, *Linear Algebra Appl.* 169 (1992) 75–101.
- [8] M.W. Hirsch, *Differential Topology*, Springer-Verlag, New York, NY, 1976.
- [9] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, Amsterdam, North Holland, 1973.
- [10] P.-F. Hsieh, Y. Sibuya, A global analysis of matrices of functions of several variables, *J. Math. Anal. Appl.* 14 (1966) 332–340.
- [11] A. Kriegl, P.W. Michor, *The Convenient Setting of Global Analysis*, American Mathematical Society, Providence, RI, 1997.
- [12] J.W. Milnor, J.D. Stasheff, *Characteristic Classes*, Princeton University Press, Princeton, NJ, 1974.
- [13] P. Sen, M.R. Chidambara, Doležal’s theorem revisited, *Math. Systems Theory* 13 (1979/80) 67–79.
- [14] Y. Sibuya, Some global properties of matrices of functions of one variable, *Math. Annalen* 161 (1965) 67–77.
- [15] L.M. Silverman, R.S. Bucy, Generalizations of a theorem of Doležal, *Math. Systems Theory* 4 (1970) 334–339.
- [16] N. Steenrod, *The Topology of Fibre Bundles*, Princeton University Press, Princeton, 1951.
- [17] L. Weiss, P.L. Falb, Doležal’s theorem, linear algebra with continuously parametrized elements, and time-varying systems, *Math. Systems Theory* 3 (1969) 67–75.
- [18] R.O. Wells, *Differential Analysis on Complex Manifolds*, Prentice Hall, Englewood Cliffs, NJ, 1973.
- [19] J.H.C. Whitehead, A certain region in Euclidean 3-space, *Proc. Nat. Acad. Sci.* 21 (1935) 364–366.